Foundations and Applications of Modal Type Theories

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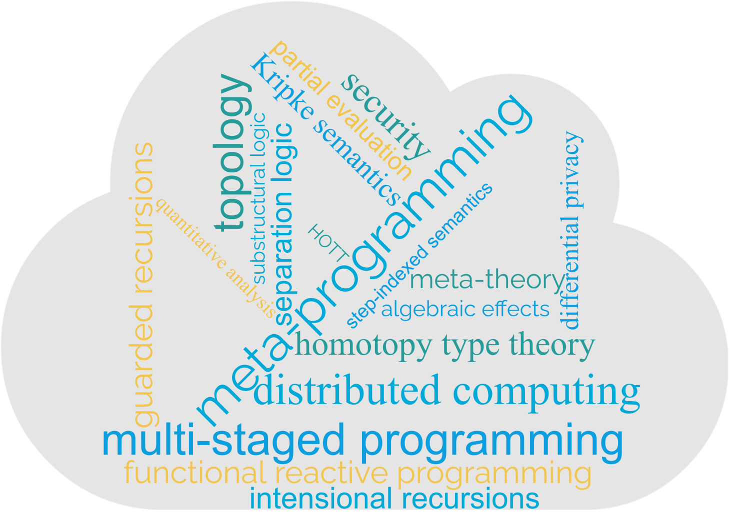
Proposal Examination

Applications of Modalities



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  - The Kripke style (Davies and Pfenning, 2001; Pfenning and Wong, 1995)
- Foundations of $\square$ still under active investigations (Hu and Pientka, 2022; Valliappan et al., 2022; Gratzer et al., 2019; Gratzer, 2022, etc.).

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- ✓ Extend $\lambda^{\to\square}$ with dependent types, yielding $\textsc{Mint}$ (submitted to JFP)
- ✓ Mechanize normalization proofs of these systems (submitted to JFP)
- ▶ Add pattern matching on code to modal type theory to strengthen its ability to do dependently typed meta-programming

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- Challenge: separate reasoning of substitutions and modal transformations leads to complex analyses

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- Enable a formulation of contextual types (Nanevski et al., 2008) in the Kripke style

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  - the algorithm can be run in Haskell after extraction

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- Internal analysis of syntactic structure:

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- Possible implementation after PhD

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