Foundations and Applications of Modal Type Theories

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PhD Defense

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Outline

- Motivations

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  - Type theory: what and why?

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  - DELAM: Recursion on syntactic objects

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  - Type theory: what and why?
  - Problem: how to extend type theory with meta-programming?
- Contributions: meta-programming in type theory
  - MINT: Quasi-quotation
  - DELAM: Recursion on syntactic objects
- Conclusions

Type Theory: What and Why

- Theoretic foundation of popular proof assistants (Coq, Agda, Lean)

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Type Theory: What and Why

- Theoretic foundation of popular proof assistants (Coq, Agda, Lean)
    - CompCert, CertikOS
    - 4 color theorem, mathlib
- Easy to understand and implement
- Propositions-as-types: same language for programming and proving

Programming and Proving in Type Theory

- Type theory is a programming language!

```
mult : Nat → Nat → Nat
mult m          n = ?
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- Type theory knows `mult 1 m` ≈ `mult 0 m + m` ≈ `0 + m` ≈ `m`

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right-id-mult : ∀ (m : Nat) → mult m 1 ≡ m
right-id-mult zero     = refl -- 0 ≡ 0
right-id-mult (succ m) = begin
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- Solution: meta-programming, i.e. write programs to generate programs and proofs

Research question:

can a type theory directly support meta-programming?

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untyped

dependently typed
(this thesis)

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Mint

Part I:

- Hu and Pientka (2022), A Categorical Normalization Proof for the Modal Lambda-Calculus, MFPS'22
- Hu et al. (2023), Normalization by Evaluation for Modal Dependent Type Theory, JFP

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- Hu et al. (2023), Normalization by Evaluation for Modal Dependent Type Theory, JFP

Part II:

- Hu and Pientka (2024), Layered Modal Type Theory: Where Meta-programming Meets Intensional Analysis, ESOP'24
- Hu and Pientka (2025), A Dependent Type Theory for Meta-programming with Intensional Analysis, POPL'25

Part I

MINT and Quasi-quotation

Quasi-quotation in MINT

- Extend dependent type theory with the $\Box$ modality

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mult2 : Nat → □(Nat → Nat)
mult2 zero     = box (λ n. 0)
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- Running meta-programs:

```
unbox₀ (mult2 2) ≈ (λ n. n + n)
```

Proving in MINT

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- Soundness: evaluating `mult2` computes the same as `mult`

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sound : ∀ (m n : Nat) → (unbox₀ (mult2 m)) n ≡ mult m n
sound m        n = ?
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LHS:

$$(\mathtt{unbox}_0\ (\mathtt{mult2}\ 0))\ n \approx (\mathtt{unbox}_0\ (\mathtt{box}\ (\lambda\ n.\ 0)))\ n \approx (\lambda\ n.\ 0)\ n \approx 0$$

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RHS:

```
mult 0 n ≈ 0
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Proving in MINT

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```
sound : ∀ (m n : Nat) → (unbox_0 (mult2 m)) n ≡ mult m n
sound zero     n = refl
sound (succ m) n = ?
```

LHS:

```
(unbox_0 (mult2 0)) n ≈ (unbox_0 (box (λ n. 0))) n ≈ (λ n. 0) n ≈ 0
```

RHS:

```
mult 0 n ≈ 0
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LHS:

```
  (unbox₀ (mult2 (succ m))) n
≈ (unbox₀ (box (λ n. (unbox₁ (mult2 m)) n + n))) n
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LHS:

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   (unbox₀ (mult2 (succ m))) n
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RHS:

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   mult (succ m) n ≈ mult m n + n
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sound : ∀ (m n : Nat) → (unbox₀ (mult2 m)) n ≡ mult m n
sound zero     n = refl
sound (succ m) n = cong (_+ n) ?
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LHS:

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  (unbox₀ (mult2 (succ m))) n
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RHS:

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```
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       (unbox₀ (mult2 (succ m))) n
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sound : ∀ (m n : Nat) → (unbox₀ (mult2 m)) n ≡ mult m n
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sound (succ m) n = cong (_+ n) (sound m n)
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  (unbox₀ (mult2 (succ m))) n
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RHS:

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Applications and Limitations of MINT

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Can we support recursion on syntactic objects in a type theory?

Part II

DeLaM and Recursion on Syntactic Objects

Analysis / Recursion on Syntactic Objects

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  - If the goal is a known truth, then it's done;
  - if the goal is a conjunction, then we need to prove each component;
  - etc.
- MINT does not support this kind of analysis!
- In general, we need to do recursion on the syntactic object of the current goal.
- A *different* type theory is needed.

Layers and DELAM

- DELAM: **De**pendent **La**yered **M**odal type theory

Layers and DeLaM

- DeLaM: **De**pendent **La**yered **M**odal type theory
    - extends dependent type theory with *layers*

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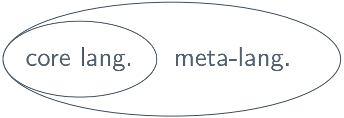

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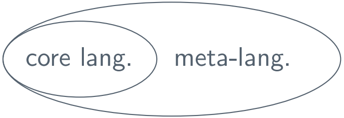

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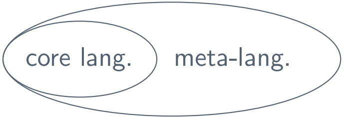

- Meta-language is an extension of core language and is strictly more expressive:
  - coherent recursion only on syntactic objects of the core language

Meta-programming in DeLaM

- Multiplication in DeLaM:

```
mult3 : Nat → □ (x : Nat ⊢ Nat)
mult3 m          = ?
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- However, these `0`'s are redundant:

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- Use `letbox` to run the generated function:

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Code Manipulation in DeLaM

- Getting rid of redundant `0`:

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- Use `simp` to simplify `0`'s away:

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mult4 : Nat → □ (x : Nat ⊢ Nat)
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Code Manipulation in DELAM

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- Finally we have the simplest forms:

```
mult4 1 ≈ box (x. x)
mult4 2 ≈ box (x. x + x)
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Proving in DeLaM

Similar to Mint, we can also prove properties about meta-programs in DeLaM

```
simp-sound : ∀ (y : □ (x : Nat ⊢ Nat)) (m : Nat) →
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simp-sound (box a)       m = ?
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Goal becomes

```
letbox s' ← simp (box (a + b)) in s'[m/x] ≡ (a + b)[m/x]
```

Also

```
simp (box (a + b)) ≈ letbox sa' ← simp (box a) in box (sa' + b)
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Goal is unblocked

```
(sa'      + b)[m/x] ≡ (a      + b)[m/x]
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- Recursion on syntactic objects:
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  - analyze and prove goals
- Run code ⇒ conflate the proving and meta-programming languages
- DELAM is a basic setup; need empirical study to understand practical needs

To Conclude

Conclusions

This PhD thesis explored ways to support meta-programming in type theory.

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This PhD thesis explored ways to support meta-programming in type theory.

- MINT supports quasi-quotation but not recursion on syntactic objects
- DELAM supports recursion on syntactic objects but mandates a less familiar programming style
- Both type theories are logically consistent and can serve as foundations for proof assistants!

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- Two properties allow to do type-checking, i.e. checking whether a program is a member of the given type

Contributions

Thesis	Part I	Part II
Type theory	MINT	DELAM
Normalization	Yes	Yes
Decidability of convertibility		
Main feature		
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Type theory	Mint	DeLaM
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A substitution replaces a variable with a term.

- $\boxed{\Gamma \vdash t \approx t' : T}$ terms t and t' are equivalent.

$$\frac{\Gamma, x : S \vdash t : T \qquad \Gamma \vdash s : S}{\Gamma \vdash (\lambda x.t)\ s \approx t[s/x] : T[s/x]}$$

- Equivalence applies to types as well.

Tactics in DeLaM

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crush g F                       = ?
```

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crush g (box (Eq Nat a b))  = ?
crush g (box (F₁ ∧ F₂))     = ?


crush g (box ((x:Nat) → F)) = ?

crush g (box _)             = ?
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                              crush g (box F₂)>>=λ (r₂ : □ (g⊢F₂)).
  letbox pf₁ ← r₁ ; pf₂ ← r₂ in Some (box (pf₁, pf₂))
crush g (box ((x:Nat) → F)) = ?

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    letbox pf₁ ← r₁ ; pf₂ ← r₂ in Some (box (pf₁, pf₂))
  crush g (box ((x:Nat) → F)) = crush (g, x:Nat) (box F)>>=
    λ (r : □ (g,x:Nat⊢F)). letbox pf ← r in Some (box (λ x. pf))
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crush g (box (Eq Nat a b))  = nat-eq-solve g (box a) (box b)
crush g (box (F1 ∧ F2))     = crush g (box F1)>>=λ (r1 : □ (g⊢F1)).
                              crush g (box F2)>>=λ (r2 : □ (g⊢F2)).
  letbox pf1 ← r1 ; pf2 ← r2 in Some (box (pf1, pf2))
crush g (box ((x:Nat) → F)) = crush (g, x:Nat) (box F)>>=
  λ (r : □ (g,x:Nat⊢F)). letbox pf ← r in Some (box (λ x. pf))
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  crush g (box _)             = ?
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crush g (box ((x:Nat) → F)) = crush (g, x:Nat) (box F)>>=
  λ (r : □ (g,x:Nat⊢F)). letbox pf ← r in Some (box (λ x. pf))
crush g (box _)             = None
```

Invoking Tactics in DeLaM

- Use tactics to avoid tedious proving steps

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  ```
  crush : (g : Ctx) ⇒ (F : □ (g ⊢ @0)) →
    letbox F' ← F in Option (□ (g ⊢ F'))

  lem : (x y : Nat) → Eq Nat (x + y) (y + x) ∧
                 ((z : Nat) → Eq Nat (x + (y + z)) (z + (y + x)))
  lem =
    let Some pf ← crush ()
      (box ((x y : Nat) → Eq Nat (x + y) (y + x) ∧
              ((z : Nat) → Eq Nat (x + (y + z)) (z + (y + x)))))
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