Layered Modal Type Theory

Where Meta-programming Meets Intensional Analysis

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at ESOP 2024

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- a type theory: normalization algorithm and proof

A Quick Comparison

System	quotation	intensional analysis	running	normalization
reflection, instrumentation	✓	✓	✓	
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Our Work	✓	✓	✓	✓

We focus on simple types as a stepping stone

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- A normalization algorithm and its completeness and soundness proof

Multiplication: An Example for Meta-programming

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```
mult : Nat → □ (x : Nat ⊢ Nat)
mult zero     = box (x. 0)
mult (succ n) = letbox u ← mult n in box (x. u[x/x] + x)
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  letbox u ← mult 2 in u[5/x] ≈ 10
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- Use `simp` to simplify `0`'s away:

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mult' : Nat → □ (Nat → Nat)
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- Finally we have the simplest forms:

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  mult' 1 ≈ box (λ x. x)
  mult' 2 ≈ box (λ x. x + x)
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Secret Ingredient: Layered Typing Judgment

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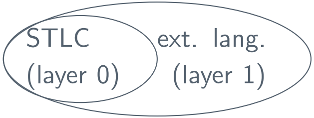

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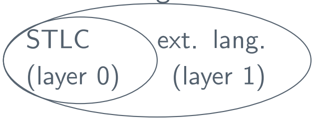
- important lemmas: static code and lifting



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Lemma (Static Code)

If $\Psi; \Gamma \vdash_0 t \approx s : T$, then $t = s$.

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- All β and η rules only occur at layer 1 $\Longrightarrow$ static code lemma

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Layering Lifts Code to Programs

Lifting lemma: characterization of layering

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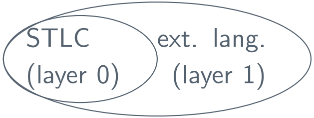

Lifting Enables Running Code

`letbox` composes and runs $\Box(\Delta \vdash T)$:

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Remark: computation is only suspended at layer 0, not lost!

```
  letbox u ← mult 2 in λ y. u[y/x]
≈ letbox u ← box (x. (0 + x) + x) in λ y. u[y/x]
≈ λ y. (0 + y) + y     -- lifting occurs
≈ λ y. y + y
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`(0 + x) + x` is frozen in `box` but eventually computes.

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What We Want (WWW)

- ✓ quotation: internally represent syntax (`box`);
- ✓ intensional analysis: *covering* pattern matching on code;
- ✓ running: evaluate a code of type A and obtain an A;
- a type theory: normalization algorithm and proof

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- The algorithm is implemented in Agda

What We Have Achieved

- ✓ quotation: internally represent syntax (`box`);
- ✓ intensional analysis: *covering* pattern matching on code;
- ✓ running: evaluate a code of type A and obtain an A;
- ✓ a type theory: normalization algorithm and proof

Takeaways

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- Layering is key to enable both code running and pattern matching on code in a coherent type theory
- A complete and sound normalization algorithm based on a presheaf model
- Our paper gives three possible future directions; reach out if you are interested!
  - System F and MLTT
  - Extending operations based on rewrite rules
  - *n* layers

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Comparison of Expressive Power

Without pattern matching, layered modal type theory is strictly weaker than $\lambda^{\square}$.

$$\begin{aligned}\mathsf{lift} &: \mathtt{Nat} \to \square\mathtt{Nat}\\ \mathsf{lift}(\mathsf{zero}) &:= \mathtt{box}\ \mathsf{zero}\\ \mathsf{lift}(\mathsf{succ}\ x) &:= \mathtt{letbox}\ u \leftarrow \mathsf{lift}(x)\ \mathtt{in}\ \mathtt{box}\ (\mathsf{succ}\ u)\end{aligned}$$

supported by both systems; turn succ $(\cdots(\mathsf{succ\ zero}))$ into box $(\mathsf{succ}\ (\cdots(\mathsf{succ\ zero})))$

$$\begin{aligned}\mathsf{nest} &: \mathtt{Nat} \to \square\mathtt{Nat}\\ \mathsf{nest}(\mathsf{zero}) &:= \mathtt{box}\ \mathsf{zero}\\ \mathsf{nest}(\mathsf{succ}\ x) &:= \mathtt{letbox}\ u \leftarrow \mathsf{nest}(x)\ \mathtt{in}\ \mathtt{box}\ (\mathtt{letbox}\ u' \leftarrow \mathsf{nest}(u)\ \mathtt{in}\ u')\end{aligned}$$

supported only by $\lambda^{\square}$ because m varies:

$$\mathsf{nest}(m) \approx \mathtt{box}\ (\mathtt{letbox}\ u_m \leftarrow \mathtt{box}\ (\cdots(\mathtt{letbox}\ u_1 \leftarrow \mathtt{box}\ \mathsf{zero}\ \mathtt{in}\ u_1)\cdots)\ \mathtt{in}\ u_m)$$

n-layered Generalization

$$\Gamma_{n-1}; \cdots; \Gamma_1; \Gamma_0 \vdash_i t : T \qquad \text{or} \qquad \overrightarrow{\Gamma} \vdash_i t : T \qquad \text{where } i \in [0, n-1].$$

Lemma (Static code)

If $i \in [0, n-2]$ and $\Psi; \Gamma \vdash_i t \approx s : T$, then $t = s$.

Lemma (Lifting)

If $\overrightarrow{\Gamma} \vdash_i t : T$ and $0 \leq i \leq j < n$, then $\overrightarrow{\Gamma} \vdash_j t : T$.