Normalization by Evaluation for Modal Dependent Type Theory

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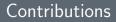
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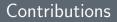
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- means to explore meta-programming in dependent type theory







► MINT, a modal dependent type theory





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- A complete and sound normalization-by-evaluation (NbE) algorithm based on an untyped domain
- \blacktriangleright A full mechanization in Agda (\sim 11k LoC) available online



```
\begin{array}{ccc} \texttt{lift} & : & \texttt{Nat} \to \Box & \texttt{Nat} \\ \texttt{lift} & \texttt{n} & & = & \red{eq: non-states} \end{array}
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lift : Nat $\rightarrow \Box$ Nat lift zero = box zero lift (succ n) = ?



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lift (succ n) = box (succ (unbox<sub>1</sub> (lift n)))
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lift n lifts into a box; returns box n

Soundness: evaluating lift gives the same number back

unbox-lift : (n : Nat) \rightarrow unbox₀ (lift n) \equiv n unbox-lift n = ?



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► Base case:

 $unbox_0$ (box zero) \equiv zero





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► After congruence,

 $(unbox_0 (lift n)) \equiv n$



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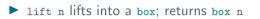
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MINT as a program logic for MetaML



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 $\frac{x:T\in\Gamma}{\overrightarrow{\Gamma};\Gamma\vdash x:T}$



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Modal extension \overrightarrow{l} ; \Uparrow^n is a special substitution which fixes the context stack.



Handling Context Stack with Substitutions



► A modal extension adds contexts to the domain stack (*n* could be 0)

$$\frac{\overrightarrow{\Gamma} \vdash \overrightarrow{\sigma} : \overrightarrow{\Delta} \qquad |\overrightarrow{\Gamma}'| = n}{(\overrightarrow{\Gamma}; \overrightarrow{\Gamma}') \vdash (\overrightarrow{\sigma}; \uparrow^n) : (\overrightarrow{\Delta}; \cdot)}$$

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 $\blacktriangleright \Box T \text{ is } \eta \text{ expandable}$

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- ► Two steps:
 - 1. evaluation: evaluate a well-typed term into a domain value and eliminate all β redexes
 - 2. readback: read from a domain value back to a normal form and perform type-directed η expansion during the process.



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- \blacktriangleright The normalization algorithm and its completeness and soundness theorems are mechanized in Agda (\sim 11k LoC)



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- A different flavor: layered modal type theory: simple types (Hu and Pientka, 2024b); dependent types (Hu and Pientka, 2024a)
- Modal type theories have solutions in different flavors





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Theorem (Completeness)

If
$$\overrightarrow{\Gamma} \vdash t \approx t'$$
: *T*, then $\mathsf{nbe}_{\overrightarrow{\Gamma}}^{T}(t) = \mathsf{nbe}_{\overrightarrow{\Gamma}}^{T}(t')$.

Theorem (Soundness)

If
$$\overrightarrow{\Gamma} \vdash t : T$$
, then $\overrightarrow{\Gamma} \vdash t \approx \mathsf{nbe}_{\overrightarrow{\Gamma}}^{T}(t) : T$.

Theorem (Consistency)

There is no closed term of type $\Pi(x : Ty_i).x$.

Jason Hu, Junyoung Jang, Brigitte Pientka — Normalization by Evaluation for Modal Dependent Type Theory